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### ► To cite this version:

Nicolas Ratier, Rémi Brendel, Philippe Guillemot. Quartz oscillators: deriving oscillation condition by symbolic calculus. Proc. of the Tenth European Frequency and Time Forum, Mar 1996, Brighton, United Kingdom. pp.442-446. hal-00345153

**HAL Id: hal-00345153**

**<https://hal.science/hal-00345153>**

Submitted on 8 Dec 2008

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# QUARTZ OSCILLATORS: DERIVING OSCILLATION CONDITION BY SYMBOLIC CALCULUS

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## Abstract

This paper presents the method used to derive the oscillation condition by using symbolic calculus. The program is based on the full nonlinear Barkhausen criterion method. The behaviour of an oscillator is described by a complex polynomial called the characteristic polynomial. This polynomial enables us to calculate the steady state features of the oscillation as well as the differential equation for transient analysis in the time domain.

The literal determination of this characteristic polynomial involves lengthy algebraic calculations and cannot be done by hand as soon as the electronic oscillator circuit involves too many components. We recently developed a formal calculus program allowing to automatically obtain all necessary equations for oscillation analysis. We propose new methods to calculate them in an optimal form.

## Introduction

Accurate analysis of an actual oscillator behaviour requires heavy calculus even for the simplest oscillator structure. Designers need automatic tools to determine oscillation frequency, amplitude and sensitivity to parameters change. We developed such a simulation tool dedicated to the modeling of quartz oscillators.

This paper presents the actual state of a computer program especially designed to accurately simulate the behaviour of quartz crystal oscillators. The plan of the paper is as follows. The first section recalls the method employed to derive the oscillation condition. Reader should consult [3][2][1] where all the explanation are carried out with complete detail. The second section describes the main problem arising when the oscillation condition is computed by symbolic calculus. The next two sections are devoted to the solution of this problem.

## Oscillation condition derivation

The transistor is modelled by a large signal admittance parameter two-port circuit. The  $y$ -parameters of which have a real and an imaginary part both being nonlinear functions of the signal amplitude (Fig. 1)(Eq.1).

$$y_n = g_n + sC_n \text{ with } n = i, r, f, o \quad (1)$$

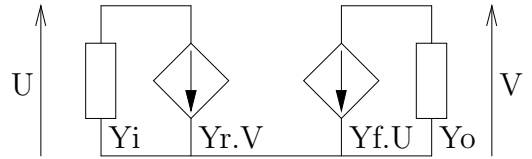


Figure 1:  $y$ -parameter representation of a transistor

The numerical values of the  $y$ -parameter are calculated at run time by using the electrical simulator SPICE [5][7]. These values depend on the amplitude and, of course, on bias and temperature conditions [3].

A quartz crystal oscillator can be reduced to the canonical form represented in Fig. 2 by using a few number of transformations [6].

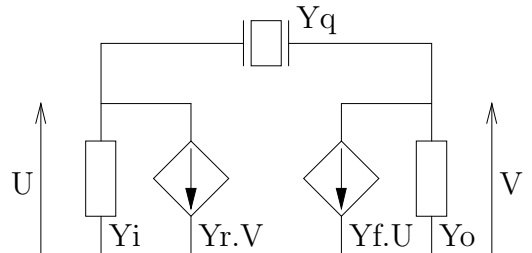


Figure 2: Reduced form of an oscillator

These transformations are divided into three groups. The first one is composed of elementary transforma-

tions on the circuit admittances like series, parallel combinations or Kennelly's transforms. Figure 3 shows one of the elementary transformation.

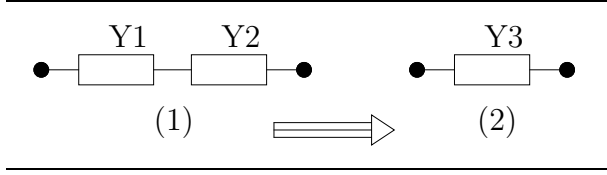


Figure 3: Elementary transformation of the circuit

Equations 2 represents the associated relationship.

$$\begin{aligned} Y_1 &\triangleq \frac{N_1}{D_1} \\ Y_2 &\triangleq \frac{N_2}{D_2} \\ Y_3 &= \frac{N_1 N_2}{N_1 D_2 + D_1 N_2} \end{aligned} \quad (2)$$

The second group is made up by transformations involving the  $y$ -parameter equivalent circuit and the admittances around it. Figure 4 represents an example of these transformations.

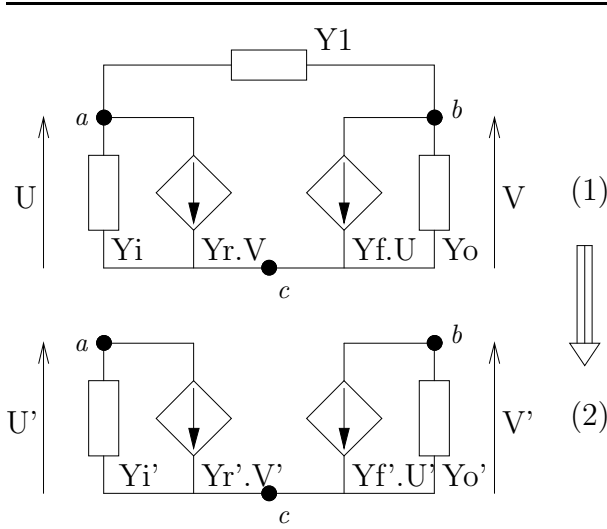


Figure 4: Elementary transformation of the circuit

After such a reduction, the components of the new  $y$ -parameter equivalent circuit are expressed in terms of the initial circuit component, as shown by (3).

$$\begin{aligned} Y_i' &\triangleq \frac{N_i'}{D_i'} = \frac{N_i D_1 + D_i N_1}{D_i D_1} \\ Y_r' &\triangleq \frac{N_r'}{D_r'} = \frac{N_r D_1 - D_r N_1}{D_r D_1} \\ Y_f' &\triangleq \frac{N_f'}{D_f'} = \frac{N_f D_1 - D_f N_1}{D_f D_1} \\ Y_o' &\triangleq \frac{N_o'}{D_o'} = \frac{N_o D_1 + D_o N_1}{D_o D_1} \end{aligned} \quad (3)$$

$$\begin{aligned} U' &= U \\ V' &= V \end{aligned}$$

The third and last group involves only the  $y$ -parameter equivalent circuit. If the quartz is not connected between the base and the collector of the  $y$ -parameter equivalent circuit, it is necessary to swap two of the terminals of the amplifier. Figure 5 illustrates that kind of transformation.

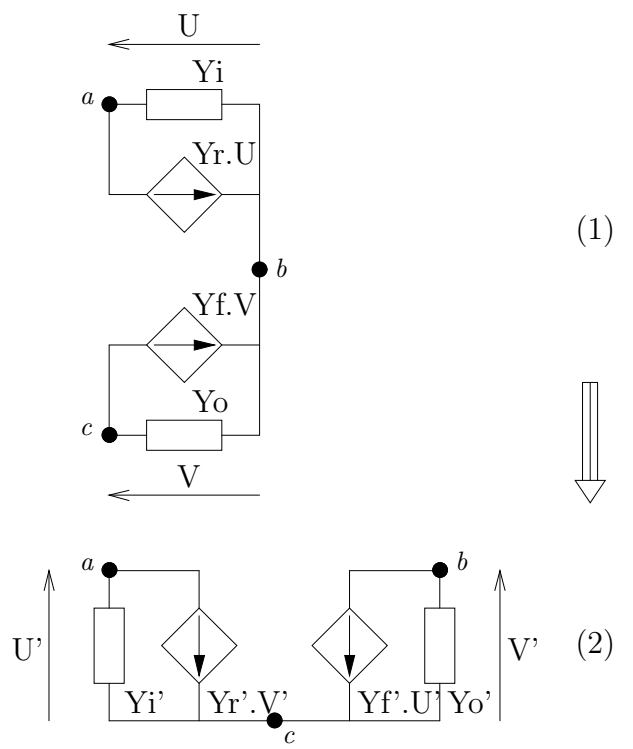


Figure 5: Elementary transformation of the circuit

The representative equations of this transformation are given by (4).

$$\begin{aligned} Y_i' &\triangleq \frac{N_i'}{D_i'} = \frac{N_i}{D_i} \\ Y_r' &\triangleq \frac{N_r'}{D_r'} = \frac{-N_i D_r - D_i N_r}{D_i D_r} \\ Y_f' &\triangleq \frac{N_f'}{D_f'} = \frac{-N_i D_f - D_i N_f}{D_i D_f} \\ Y_o' &\triangleq \frac{N_o'}{D_o'} = \frac{N_i D_r D_f D_o + \dots + D_i D_r D_f N_o}{D_i D_r D_f D_o} \\ U' &= U - V \\ V' &= -V \end{aligned} \quad (4)$$

All the transformations described above are repeated until the final reduced form shown in figure 2 is obtained. Under this form, the oscillation condition can be written:

$$Y_i Y_o - Y_f Y_r + (Y_i + Y_r + Y_f + Y_o) Y_q = 0 \quad (5)$$

In terms of the numerator  $N_\alpha$  and denominator  $D_\alpha$  of the admittance  $Y_\alpha$ , the oscillation condition takes the following form:

$$\begin{aligned} & (N_i D_r D_f N_o - D_i N_r N_f D_o) D_q \\ & + (N_i D_r D_f D_o + D_i N_r D_f D_o \\ & + D_i D_r N_f D_o + D_i D_r D_f N_o) N_q = 0 \end{aligned} \quad (6)$$

Each term of the relation is expressed under the form of a function of previous relation until reaching the first coefficients which are expressed in terms of the circuit elements  $R_\alpha, L_\alpha, C_\alpha$ . The polynomial (6) has the general form (7). Each coefficient  $a_k$  of the polynomial is expressed as a function of the component value of the circuit

$$\sum_{k=0}^K a_k s^k = 0 \quad (7)$$

When the characteristic polynomial coefficients are defined, it is possible to obtain the steady state frequency and amplitude of the oscillation. To this end, the Laplace's variable  $s$  is changed into the harmonic variable  $j\omega$ , this splits equation (7) into two equations, one for the real part (Eq. 8), the other one for the imaginary part (Eq. 9).

$$\sum_{i=0}^N \alpha_i(u) \omega^i = 0 \quad (8)$$

$$\sum_{i=0}^M \beta_i(u) \omega^i = 0 \quad (9)$$

The numerical calculation of the variables  $u$  et  $\omega$  which satisfy both equations (8, 9) determines the frequency and the amplitude of the oscillation. This resolution uses data calculated by the SPICE program and needs the determination of the first derivatives of equations (8, 9). The numerical zero finding algorithm used to solve these equations is described in [1].

Because the characteristic polynomial coefficients are expressed as functions of all the circuit components, it is possible for the program to calculate the influence of a change of any component value on the oscillation amplitude and frequency as well as on the resonator excitation level. In the same way, it is possible to get the sensitivity of these quantities to small component variation.

In addition of the rated value the user can specify the tolerance and the temperature coefficient of each component in the input data file. This allows the program to calculate the induced dispersion and to perform worst case analysis. The temperature behaviour of the resonator is described by the four coefficients of the frequency temperature curve cubic regression so that the frequency variation can be calculated over a given temperature range.

## Setting the problem

After the circuit is read, the numerator and the denominator of each admittance of the circuit is given a name. That define the first coefficients.

$$\begin{aligned} B_1 &= s C_1 \\ B_2 &= 1 \\ \dots &= \dots \\ B_{2l} &= R_4 \end{aligned} \quad (10)$$

During the reduction process, expression of the new admittances are generated as a function of the previous coefficients.

$$\begin{aligned} B_{2l+1} &= \mathbf{B}_{2l+1}(B_1, B_2, \dots, B_{2l}) = B_1 B_4 + B_2 B_3 \\ B_{2l+2} &= \mathbf{B}_{2l+2}(B_1, B_2, \dots, B_{2l+1}) = B_2 B_4 \\ \dots &= \dots \\ B_{2n} &= \mathbf{B}_{2n}(B_1, B_2, \dots, B_{2n-1}) = \dots \end{aligned} \quad (11)$$

When all the reductions have been performed, the equivalent circuit has the reduced form shown in figure 2. The oscillation condition gives the last coefficient:

$$B_{2n+1} = (B_{81} B_{84} B_{86} B_{87} - B_{82} B_{83} B_{85} B_{88}) B_{90} + \dots \quad (12)$$

The equation  $B_{2n+1} = 0$  can be written in the form of a polynomial of the Laplace's variable  $s$ :

$$\sum_{k=0}^K a_k s^k = 0 \quad (13)$$

The main problem is to compute the four following expressions from the coefficients  $B_1, B_2, \dots, B_{2n+1}$ .

$$\begin{aligned} P &= \sum_{i=0}^N \alpha_i(u) \omega^i \quad \left| \quad \frac{\partial P}{\partial \omega} \right. \\ Q &= \sum_{i=0}^M \beta_i(u) \omega^i \quad \left| \quad \frac{\partial Q}{\partial \omega} \right. \end{aligned}$$

The easiest way is to substitute recursively each  $B_i$  by their expression in the last coefficient  $B_{2n+1}$ . By this method, we obtain the polynomial  $\sum_{k=0}^K a_k s^k = 0$  where each coefficient is expressed as a function of the component of the circuit  $R_\alpha, L_\alpha, C_\alpha$ . Unfortunately, the full development of the polynomial coefficient leads to too intricate expression even in simple oscillator structures.

A better way is to express the coefficients of an admittance newly defined as a function of the coefficient previously defined. This method is detailed in the following example. Suppose that the admittance  $Y_3$  is generated after the parallel reduction of the admittance  $Y_1$  and  $Y_2$ .

$$\begin{aligned} Y_3 &= Y_1 + Y_2 \\ &= \frac{\dots}{a_2 s^2 + a_1 s + a_0} + \frac{\dots}{b_3 s^3 + b_2 s^2 + b_1 s + b_0} \\ &= \frac{\dots}{c_5 s^5 + c_4 s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0} \end{aligned} \quad (14)$$

This reduction process involve the definition of six new coefficients  $c_0, c_1, \dots, c_5$  (Eq. 15).

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_1 b_0 + a_0 b_1 \\ c_2 &= a_2 b_0 + a_1 b_1 + a_0 b_2 \\ c_3 &= a_2 b_1 + a_1 b_2 + a_0 b_3 \\ c_4 &= a_2 b_2 + a_1 b_3 \\ c_5 &= a_2 b_3 \end{aligned} \quad (15)$$

This method has the following major drawback: The degree of the polynomial increases at each reduction step. The consequence is that the number of new coefficients defined during one step increases with the number of transformations already done. This exponential increase limits the complexity of the oscillator circuit which can be treated.

The method proposed here will computes in the same time the real and the imaginary part of the equation (7 or 13). The first derivatives  $\partial P/\partial\omega, \partial Q/\partial\omega$  are calculated from these coefficients. These values are calculated during the same step. This method will be further explained in the next two sections. The advantages over the previous method are:

- The number of coefficients increases linearly with the number of reduction steps.
- These coefficient are coded in a efficient way.
- In consequence of it, the numerical calculation of the four functions  $P, Q, \partial P/\partial\omega, \partial Q/\partial\omega$  is almost optimal.

Moreover, this method allows one to calculate efficiently the values of  $P, Q, \partial P/\partial\omega, \partial Q/\partial\omega$  without knowing explicitly the expression of the coefficients  $a_k$  in equation (7).

## Determination of the real and imaginary part

The first coefficients defined (see Eq. 10) are function of the Laplace's variable  $s$ , so the splitting into real and imaginary part is straightforward. Equations (11) and (12) show that all the other equations have the specific form of equation (16). In this particular case, an efficient method to calculate symbolically the real and imaginary part of the coefficient can be used.

$$B_{i+1} = \sum \prod_{k \leq i} B_k \quad (16)$$

Suppose that the real and the imaginary part of each coefficient  $B_\alpha$  is known up to the  $i$ th order.

$$\Re(B_1), \Re(B_2), \dots, \Re(B_i), \quad (17)$$

$$\Im(B_1), \Im(B_2), \dots, \Im(B_i) \text{ are known} \quad (18)$$

It is possible to calculate the real and the imaginary part of the coefficient  $B_{i+1}$ . We shall illustrate the method on a concrete example. Consider the expression

$$Exp = B_1 B_2 B_3 + B_4 B_5 \quad (19)$$

The method introduces new coefficient  $C_\alpha$ . These coefficient are calculated so that only one complex multiplication is done in each step. In the case of the previous expression, the result is:

$$\begin{aligned} C_1 &= \Re(B_1)\Re(B_2) - \Im(B_1)\Im(B_2) \\ C_2 &= \Re(B_1)\Im(B_2) + \Im(B_1)\Re(B_2) \\ C_3 &= C_1\Re(B_3) - C_2\Im(B_3) \\ C_4 &= C_1\Im(B_3) + C_2\Re(B_3) \\ C_5 &= \Re(B_4)\Re(B_5) - \Im(B_4)\Im(B_5) \\ C_6 &= \Re(B_4)\Im(B_5) + \Im(B_4)\Re(B_5) \\ \Re(Exp) &= C_3 + C_5 \\ \Im(Exp) &= C_4 + C_6 \end{aligned} \quad (20)$$

Notice that the real part and the imaginary part are computed in the same time. Let us compare this method with the more direct one.

$$\begin{aligned} \Re(Exp) &= \Re(B_1)\Re(B_2)\Re(B_3) - \Re(B_1)\Im(B_2)\Im(B_3) \\ &\quad - \Im(B_1)\Re(B_2)\Im(B_3) - \Im(B_1)\Im(B_2)\Re(B_3) \\ &\quad + \Re(B_4)\Re(B_5) - \Im(B_4)\Im(B_5) \\ \Im(Exp) &= \Re(B_1)\Re(B_2)\Im(B_3) + \Re(B_1)\Im(B_2)\Re(B_3) \\ &\quad + \Im(B_1)\Re(B_2)\Re(B_3) - \Im(B_1)\Im(B_2)\Im(B_3) \\ &\quad + \Re(B_4)\Im(B_5) + \Im(B_4)\Re(B_5) \end{aligned} \quad (21)$$

Table 1 indicates the number of arithmetic operation + and \* needed to express the form  $B_1 B_2 \dots B_i$ . The two methods are compared for expressions of different lengths.

	Direct		Proposed	
	Nb +	Nb *	Nb +	Nb *
$B_1$	0	0	0	0
$B_1 B_2$	2	4	2	4
$B_1 B_2 B_3$	6	16	4	8
$B_1 B_2 B_3 B_4$	14	48	6	12
$B_1 B_2 B_3 B_4 B_5$	30	128	8	16

Table 1: Computation cost

## Determination of the first derivatives

The first coefficients are necessarily defined as a function of  $\omega$  and of the admittance of the circuit. So their derivatives respect to  $\omega$  can be easily calculated. Now suppose that the first derivative of each coefficient  $C_\alpha$  is known up to the  $i$ th order.

$$\frac{\partial C_1}{\partial \omega}, \frac{\partial C_2}{\partial \omega}, \dots, \frac{\partial C_i}{\partial \omega} \quad (23)$$

It is possible to calculate the derivative of the coefficient  $C_{i+1}$  by expanding as follows.

$$\frac{\partial C_{i+1}}{\partial \omega} = \frac{\partial C_{i+1}}{\partial C_1} \frac{\partial C_1}{\partial \omega} + \dots + \frac{\partial C_{i+1}}{\partial C_i} \frac{\partial C_i}{\partial \omega} \quad (24)$$

One might think that the form of the  $\partial C_i / \partial \omega$  will be extremely complicated. It is not the case because of the specificity of our problem. From equations (20), it easily follows that each coefficient  $C_\alpha$  depends only on four other coefficients. The consequence is that the expression of the derivative will be quite simple. We showed above an example. The expression of the derivative  $D_{10}$  of the coefficient  $C_{10}$  is given by (26), where all the coefficients  $C_1, \dots, C_9$  and  $D_1, \dots, D_9$  are known.

$$C_{10} = C_3 C_5 - C_4 C_6 \quad (25)$$

$$D_{10} = D_3 C_5 + C_3 D_5 - D_4 C_6 - C_4 D_6 \quad (26)$$

## Conclusion

The works presented here highlights some problems related to the complexity of the expressions to be manipulated. The proposed methods allow an optimization of the calculus at the level of the topological analysis as well as for numerical calculation of the required coefficients to the calculation of the oscillation's condition.

The fact that this optimization is not a simple calculation refinement that one can do without should be stressed. The present method is necessary in order to treat complex oscillator circuits. Indeed, we have shown that with the previous method the complexity of the calculus exponentially increases as a function of the circuit components while the complexity of the new method linearly increases.

## Acknowledgements

This work has been supported by CNES and DGA under contracts: # 962/CNES/91/1476/00, # 962/CNES/94/1230/00.

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